

**Names:** Russell Bishop, Raymond Bower & Robert Maholic

**Grade Level:** 9 – 12

**Content Area:** Algebra

**PA Standard(s) addressed:**

2.8.11.a,b,c – Algebra and Functions / Patterns

**NCTM Standard(s) addressed:**

Algebra and Functions



**Problem Name:** Making Change

**Problem:**

When Bernie was 12 years old he used to deliver newspapers. His mom gave him a coin holder that did not hold quarters.

(a) If the Sunday paper cost 50¢ and Bernie was given a \$1.00 bill, how many different ways could he make change? (Remember no quarters)



(b) For a one month promotion the price was dropped to 40¢. Now when Bernie was given a \$1.00 bill, how many different ways could he make change? (Still no quarters)

(c) Form a generalization to determine the number of ways little Bernie could make change for any multiple of 10¢ using only dimes, nickels and pennies.

**Directions:**

1. Find a correct solution for each part. Show all work used to determine the solution indicating where any work was done using a calculator or mental math.
2. Explain the reasoning or process used to arrive at each solution.

**Extension:**

- (d) For his 13<sup>th</sup> birthday Bernie's mom gave him a new coin holder that also held quarters. Now how many different ways could Bernie make change for \$1.00?

**Problem Solution(s):**

**Solution for Question (a)**

<b>dimes</b>	<b>nickels</b>	<b>pennies</b>	<b>dime group</b>
5	0	0	1
4	2	0	
4	1	5	3
4	0	10	
3	4	0	
3	3	5	
3	2	10	5
3	1	15	
3	0	20	
2	6	0	
2	5	5	
2	4	10	
2	3	15	7
2	2	20	
2	1	25	
2	0	30	
1	8	0	
1	7	5	
1	6	10	
1	5	15	
1	4	20	9
1	3	25	
1	2	30	
1	1	35	
1	0	40	
0	10	0	
0	9	5	
0	8	10	
0	7	15	
0	6	20	
0	5	25	11
0	4	30	
0	3	35	
0	2	40	
0	1	45	
0	0	50	
<b>total</b>			<b>36</b>

This table represents all the ways of making change for \$0.50  
 The numbers in the right column represent the total for each grouping of dimes.  
 Therefore there are 36 ways to make change.

The table shows all the possible ways to make change using dimes, nickels and pennies.

**Solution for Question (b)**

<b>dimes</b>	<b>nickels</b>	<b>pennies</b>	<b>dime group</b>
6	0	0	1
5	2	0	
5	1	5	3
5	0	10	
4	4	0	
4	3	5	
	⋮		5
4	0	20	
3	6	0	
	⋮		7
3	0	30	
2	8	0	
	⋮		9
2	0	40	
1	10	0	
	⋮		11
1	0	50	
0	12	0	
	⋮		13
0	0	60	
<b>total</b>			<b>49</b>

This table represents all the ways of making change for \$0.60  
 The numbers in the right column represent the total for each grouping of dimes.  
 Therefore there are 49 ways to make change.  
 The blank rows in the table indicate the pattern continues.

### Solution for Question (c)

After recognizing a pattern in parts (a) and (b), one can generalize that the number of ways of making change for any amount of money that is a multiple of 10 cents using only dimes, nickels, and pennies is equal to the sum of an arithmetic series. Following this reasoning, one is able to arrive at an explicit formula for the number of ways of making change.

Let  $A$  = amount of money in cents,

$$\text{then the maximum number of dimes possible} = \frac{A}{10}$$

$$\text{and the number of terms} = \frac{A}{10} + 1$$

For example, given 40¢ it would take a maximum of  $\frac{40}{10} = 4$  dimes and the number of terms in the arithmetic sequence would equal  $\frac{40}{10} + 1 = 5$ . These five terms represent the number of ways of making the 40¢ change using 4, 3, 2, 1, or 0 dimes respectively.

The first term is always 1 since there is only 1 way to make change for amount  $A$  using all dimes. For example, there is only 1 way to make change for 40¢ using 4 dimes.

As we reduce the number of dimes being given in change we replace the value of the dimes using some other combination of coins. Converting a dime to nickels increases the maximum possible number of nickels given in change by 2. Thus if we had a maximum of  $n$  nickels before, we now have  $n+2$ . We could now give anywhere from  $n+2$  down to 0 nickels in change (as opposed to  $n$  down to 0 in the previous step) with the balance being given in pennies. Thus the constant difference in the terms is 2.

The pattern continues until we have reduced the number of dimes given to 0.

The sum of an arithmetic sequence is  $\frac{n}{2}(a_1 + a_n)$  or  $\frac{n}{2}(2a_1 + (n-1)d)$ . Thus an arithmetic sequence

with first term = 1, constant difference = 2, and number of terms =  $\frac{A}{10} + 1$ , the sum is

$$\frac{\frac{A}{10} + 1}{2} (2 + (\frac{A}{10} + 1 - 1)2) = (\frac{A}{10} + 1)^2.$$

Extension: Number of ways to change \$1

# Quarters	# Dimes	# Nickels	# Pennies		# Quarters	# Dimes	# Nickels	# Pennies		# Quarters	# Dimes	# Nickels	# Pennies		# Quarters	# Dimes	# Nickels	# Pennies		# Quarters	# Dimes	# Nickels	# Pennies	
4	0	0	0	1	1	7	1	0	2	1	1	13	0	14	0	5	10	0	11	0	1	18	0	19
					1	7	0	5		1	1	12	5		0	5	9	5		0	1	17	5	
3	2	1	0	2	1	6	3	0	4	1	1	11	10		0	5	8	10		0	1	16	10	
3	2	0	5		1	6	2	5		1	1	10	15		0	5	7	15		0	1	15	15	
3	1	3	5	4	1	6	1	10		1	1	9	20		0	5	6	20		0	1	14	20	
3	1	2	10		1	6	0	15		1	1	8	25		0	5	5	25		0	1	13	25	
3	1	1	15		1	5	5	0	6	1	1	7	30		0	5	4	30		0	1	12	30	
3	1	0	20		1	5	4	5		1	1	6	35		0	5	3	35		0	1	11	35	
3	0	5	0	6	1	5	3	10		1	1	5	40		0	5	2	40		0	1	10	40	
3	0	4	5		1	5	2	15		1	1	4	45		0	5	1	45		0	1	9	45	
3	0	3	10		1	5	1	20		1	1	3	50		0	5	0	50		0	1	8	50	
3	0	2	15		1	5	0	25		1	1	2	55		0	4	12	0	13	0	1	7	55	
3	0	1	20		1	4	7	0	8	1	1	1	60		0	4	11	5		0	1	6	60	
3	0	0	25		1	4	6	5		1	1	0	65		0	4	10	10		0	1	5	65	
				12	1	4	5	10		1	0	15	0	16	0	4	9	15		0	1	4	70	
2	5	0	0	1	1	4	4	15		1	0	14	5		0	4	8	20		0	1	3	75	
2	4	2	0	3	1	4	3	20		1	0	13	10		0	4	7	25		0	1	2	80	
2	4	1	5		1	4	2	25		1	0	12	15		0	4	6	30		0	1	1	85	
2	4	0	10		1	4	1	30		1	0	11	20		0	4	5	35		0	1	0	90	
2	3	4	0	5	1	4	0	35		1	0	10	25		0	4	4	40		0	0	20	0	21
2	3	3	5		1	3	9	0	10	1	0	9	30		0	4	3	45		0	0	19	5	
2	3	2	10		1	3	8	5		1	0	8	35		0	4	2	50		0	0	18	10	
2	3	1	15		1	3	7	10		1	0	7	40		0	4	1	55		0	0	17	15	
2	3	0	20		1	3	6	15		1	0	6	45		0	4	0	60		0	0	16	20	
2	2	6	0	7	1	3	5	20		1	0	5	50		0	3	14	0	15	0	0	15	25	
2	2	5	5		1	3	4	25		1	0	4	55		0	3	13	5		0	0	14	30	
2	2	4	10		1	3	3	30		1	0	3	60		0	3	12	10		0	0	13	35	
2	2	3	15		1	3	2	35		1	0	2	65		0	3	11	15		0	0	12	40	
2	2	2	20		1	3	1	40		1	0	1	70		0	3	10	20		0	0	11	45	
2	2	1	25		1	3	0	45		1	0	0	75		0	3	9	25		0	0	10	50	
2	2	0	30		1	2	11	0	12					72	0	3	8	30		0	0	9	55	
2	1	8	0	9	1	2	10	5		0	10	0	0	1	0	3	7	35		0	0	8	60	
2	1	7	5		1	2	9	10		0	9	2	0	3	0	3	6	40		0	0	7	65	
2	1	6	10		1	2	8	15		0	9	1	5		0	3	5	45		0	0	6	70	
2	1	5	15		1	2	7	20		0	9	0	10		0	3	4	50		0	0	5	75	
2	1	4	20		1	2	6	25		0	8	4	0	5	0	3	3	55		0	0	4	80	
2	1	3	25		1	2	5	30		0	8	3	5		0	3	2	60		0	0	3	85	
2	1	2	30		1	2	4	35		0	8	2	10		0	3	1	65		0	0	2	90	
2	1	1	35		1	2	3	40		0	8	1	15		0	3	0	70		0	0	1	95	
2	1	0	40		1	2	2	45		0	8	0	20		0	2	16	0	17	0	0	0	100	
2	0	10	0	11	1	2	1	50		0	7	6	0	7	0	2	15	5		0	0	0	100	
2	0	9	5		1	2	0	55		0	7	5	5		0	2	14	10						121
2	0	8	10							0	7	4	10		0	2	13	15						
2	0	7	15							0	7	3	15		0	2	12	20						
2	0	6	20							0	7	2	20		0	2	11	25						
2	0	5	25							0	7	1	25		0	2	10	30						
2	0	4	30							0	7	0	30		0	2	9	35						
2	0	3	35							0	6	8	0	9	0	2	8	40						
2	0	2	40							0	6	7	5		0	2	7	45						
2	0	1	45							0	6	6	10		0	2	6	50						
2	0	0	50							0	6	5	15		0	2	5	55						
				36						0	6	4	20		0	2	4	60						
										0	6	3	25		0	2	3	65						
										0	6	2	30		0	2	2	70						
										0	6	1	35		0	2	1	75						
										0	6	0	40		0	2	0	80						



**Specific Rubric:**

Total points earned on individual rubrics.

**5. Advanced Understanding:**

Total of 9 from individual rubrics

**4. Satisfactory Understanding:**

Total of 7 or 8 from individual rubrics

**3. Almost Satisfactory Understanding:**

Total of 5 or 6 from individual rubrics

**2. Partial Understanding:**

Total of 3 or 4 from individual rubrics

**1. Minimal Understanding:**

Total of 1 or 2 from individual rubrics

**0. No Understanding:**

Total of 0 from individual rubrics

**Rubric for Individual Questions:**

There is a possible of 3 points for each question.

3. - Correct solution, Sufficient work to justify answer, Complete and meaningful explanation

- 2. - Correct solution, Sufficient work to justify answer, Incomplete or missing explanation
  - Correct solution, Insufficient work to justify answer, Complete explanation
  - Incorrect solution, Sufficient work to justify answer, Meaningful explanation, may have an arithmetic or reasoning error
  
- 1. - Correct solution, Insufficient work to justify answer, Incomplete or missing explanation
  - Incorrect solution, Work or explanation that could lead to the solution
  
- 0. - Incorrect solution, No meaningful work or explanation
  - No Solution
  - Off-task solution